Holographic Dark Energy in Braneworld Models with a Gauss-Bonnet Term in the Bulk. Interacting Behavior and the $w = -1$ Crossing

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We apply bulk holographic dark energy in general braneworld models with a Gauss-Bonnet term in the bulk and an induced gravity term and a perfect fluid on the brane. Without making any additional assumptions we extract the Friedmann equation on the physical brane and we show that a ρ-ρ_Λ coupling arises naturally by the full 5D dynamics. The low-energy (late-time) evolution reveals that the effective 4D holographic dark energy behaves as "quintom", that is it crosses the phantom divide $w = -1$ during the evolution. In particular, the Gauss-Bonnet contribution decreases the present value of w_Λ, while it increases the growing rate of $w_\Lambda(z)$ with z, in comparison with the case where such a term is absent.

I. INTRODUCTION

Holographic dark energy [1, 2, 3, 4, 5, 6] is an interesting and simple idea of explaining the observed Universe acceleration [7]. It arises when the more fundamental holographic principle [8, 9] is applied in the cosmological framework [10, 11] (although there are some objections on this approach [12]). Holographic dark energy reveals the dynamical nature of the vacuum energy by relating it to cosmological volumes. The background on which it is based, is the black hole thermodynamics [13, 14] and the connection between the UV cut-of of a quantum field theory, which is related to vacuum energy, and a suitable large distance of the theory [15]. This connection, which was also known from AdS/CFT correspondence, proves to be necessary for the applicability of quantum field theory in large distances. The reason is that while the entropy of a system is proportional to its volume the black hole entropy is proportional to its area. Therefore, the total energy of a system should not exceed the mass of a black hole of the same size, since in this case the system would collapse to a black hole violating the second law of thermodynamics. When this concept is applied to the Universe, the corresponding vacuum

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energy is the holographic dark energy.

Until now, almost all works on the subject have been formulated in the standard 4D framework. On the other hand, brane cosmology [16, 17] exhibits many phenomenological successes [18]. In a recent work [19] we presented a generalized and restored holographic dark energy in the braneworld context. The basic argument was that in such a framework black holes will in general be D-dimensional [13, 14] and therefore holographic dark energy should be considered in the bulk. Subsequently, it gives rise to an effective 4D dark energy with "inherited" holographic nature, and this one is present in the (also arisen from the full dynamics) Friedmann equation of the brane. In [19] we applied this bulk holographic dark energy in a general single-brane model and we reproduced the results of conventional 4D calculations [1, 2, 3, 4, 5, 6], having in mind that the physical interpretation is different. In [20] we applied it in a general two-brane model with moving branes and we showed that "quintom" behavior [21, 22, 23] arises naturally for a large parameter space area of a simple solution subclass, without the inclusion of special fields or potential terms. In particular we found that w_Λ was larger than -1 in the past while its present value is $w_{\Lambda_0} = -1.08$, and the phantom divide $w_\Lambda = -1$ was crossed at $z_p \approx 0.49$, a result in remarkable agreement with observations [24, 25].

In this work we examine general single-brane models, including a Gauss-Bonnet term in the bulk [26, 27, 28, 29, 30, 31] (see also [32, 33] for a Gauss-Bonnet term in conventional 4D cosmology). Such a higher-curvature combination corresponds to the leading order quantum correction to gravity, in an effective action approach to string theory and in particular in the case of the heterotic string [34], and its coupling is related to the Regge slope parameter on string scale. Furthermore, the Gauss-Bonnet combination is the only curvature squared form which gives ghost-free self-interactions for the graviton (around flat spacetime) [35] and maintains its zero modes of the perturbations localized on the brane [36]. Fortunately, holographic description holds for braneworld Gauss-Bonnet gravity, although the subject is not trivial since there are some ambiguities in the case of non-flat branes away from the bulk boundary [37]. Applying bulk holographic dark energy in this framework, and without any additional assumption, we acquire the interesting situation of an interaction between the 4D dark energy and the matter density of the brane. In this case, cosmological evolution and in particular the dependence of the 4D dark energy on the brane scale factor, acquires a correction in terms of the Gauss-Bonnet coupling. The rest of the text is organized as follows: In section II we present the holographic dark energy

in the bulk and in section III we apply it to a general single-brane model in 4+1 dimensions with a Gauss-Bonnet term in the bulk. Finally, in IV we discuss the physical implications of our analysis and we summarize the obtained results.

II. FORMULATION OF HOLOGRAPHIC DARK ENERGY IN A GENERAL BULK

In this section we display the basic results of bulk holographic dark energy, formulated in [19]. The mass M_{BH} of a spherical and uncharged D-dimensional black hole is related to its Schwarzschild radius r_s through [14, 38]:

$$M_{BH} = r_s^{D-3}(\sqrt{\pi}M_D)^{D-3}M_D\frac{D-2}{8\Gamma(\frac{D-1}{2})}, \tag{2.1}$$

where the D-dimensional Planck mass M_D is related to the D-dimensional gravitational constant G_D and the usual 4-dimensional Planck mass M_p through:

$$M_D = G_D^{-\frac{1}{D-2}},$$
$$M_p^2 = M_D^{D-2}V_{D-4}, \tag{2.2}$$

with V_{D-4} the volume of the extra-dimensional space [14].

If $\rho_{\Lambda D}$ is the bulk vacuum energy, then application of holographic dark energy in the bulk gives:

$$\rho_{\Lambda D}\text{Vol}(\mathcal{S}^{D-2}) \leq r^{D-3}(\sqrt{\pi}M_D)^{D-3}M_D\frac{D-2}{8\Gamma(\frac{D-1}{2})}, \tag{2.3}$$

where $\text{Vol}(\mathcal{S}^{D-2})$ is the volume of the maximal hypersphere in a D-dimensional spacetime, given from:

$$\text{Vol}(\mathcal{S}^{D-2}) = A_D\, r^{D-1}, \tag{2.4}$$

with

$$A_D = \frac{\pi^{\frac{D-1}{2}}}{(\frac{D-1}{2})!},$$
$$A_D = \frac{(\frac{D-2}{2})!}{(D-1)!}2^{D-1}\,\pi^{\frac{D-2}{2}}, \tag{2.5}$$

for $D-1$ being even or odd respectively. Therefore, by saturating inequality (2.3) introducing L as a suitable large distance (IR cut-off) and c^2 as a numerical factor, the corresponding vacuum

energy is, as usual, viewed as holographic dark energy:

$$\rho_{\Lambda D} = c^2 (\sqrt{\pi} M_D)^{D-3} M_D A_D^{-1} \frac{D-2}{8\Gamma(\frac{D-1}{2})} L^{-2}.$$
(2.6)

As was mentioned in [19], the "suitable large distance" which is used in the definition of L in (2.6) could be the Hubble radius [39], proportional to the square root of the Hubble radius [4], the particle horizon [10], the future event horizon [1, 3, 40], or the radius of the event horizon measured on the sphere of the horizon [5] (see also [6] for the corresponding formulation in Chaplygin gas and tachyon holographic models). For a flat Universe the future event horizon is the most suitable ansatz and furthermore it is the only one that fits holographic statistical physics, namely the exclusion of those degrees of freedom of a system that will never be observed by the effective field theory [41].

III. HOLOGRAPHIC DARK ENERGY IN GENERAL 5D BRANEWORLD MODELS WITH A GAUSS-BONNET TERM IN THE BULK

We are interested in applying bulk holographic dark energy in general 5D braneworld models with a Gauss-Bonnet term in the bulk. We consider an action of the form [26, 27]:

$$S = \int d^4x dy \sqrt{-g} \left(M_5^3 R - \rho_{\Lambda 5} + M_5^3 \alpha \mathcal{L}_{GB} \right) + \int d^4x \sqrt{-\gamma} \left(\mathcal{L}_{br}^{mat} - V + r_c M_5^3 R_4 \right).$$
(3.7)

In the first integral M_5 is the 5D Planck mass, $\rho_{\Lambda 5}$ is the bulk cosmological constant which is identified as the bulk holographic dark energy, and R is the curvature scalar of the 5D bulk spacetime with metric g_{AB}. As usual,

$$\mathcal{L}_{GB} = R^2 - 4R_{AB}R^{AB} + R_{ABCD}R^{ABCD}$$
(3.8)

is the Gauss-Bonnet term with coupling constant α, and R_{ABCD}, R_{AB} are respectively the Riemann and Ricci tensors. In the second integral γ is the determinant of the induced 4D metric $\gamma_{\alpha\beta}$ on the brane, V is the brane tension and $\mathcal{L}_{br}^{mat}$ is an arbitrary brane matter content. Lastly, we have allowed for an induced gravity term on the brane, arising from radiative corrections, with r_c its characteristic length scale and R_4 the 4D curvature scalar [31, 42, 43].

In order to acquire the cosmological evolution on the brane we use the Gaussian normal coordinates with the following metric form [44, 45]:

$$ds^2 = -m^2(\tau, y)d\tau^2 + a^2(\tau, y)\, d\Omega_k^2 + dy^2.$$
(3.9)

The brane is located at $y = 0$, we impose a Z_2-symmetry around it, $m(\tau, y = 0) = 1$ and $d\Omega_k^2$ stands for the metric in a maximally symmetric 3-dimensional space with $k = -1, 0, +1$ parametrizing its spacial curvature. Although we could assume a general matter-field content [46], we consider a brane-Universe containing a perfect fluid with equation of state $p = w\rho$. In this case, and after integration of the 00 and ii components of the 5D Einstein equations around the brane, the low-energy ($\rho \ll V$) brane cosmological evolution is governed by the following equation [26, 31, 42] (see also [28] for similar brane solutions):

$$H^2 + \frac{k}{a^2} = \left(72M_5^6 - 16\alpha\,\rho_{\Lambda 5}M_5^3 + 6r_c V M_5^3\right)^{-1} V\rho + \frac{V^2}{144M_5^6} - \frac{1 - \sqrt{1 + \tilde{\Lambda}}}{36\alpha}\left(2 + \sqrt{1 + \tilde{\Lambda}}\right)^2, \tag{3.10}$$

where

$$\tilde{\Lambda} = 2\alpha\rho_{\Lambda 5}/3M_5^3. \tag{3.11}$$

In (3.10) a stands as usual for the brane scale factor. In order to acquire a form consistent with conventional 4D Friedmann equation we make the identification:

$$V = \frac{72M_5^3}{\frac{3}{8\pi}\frac{M_p^2}{M_5^3} - 6r_c}, \tag{3.12}$$

and we define

$$V_1(\alpha, \rho_{\Lambda 5}) = \frac{2\alpha\rho_{\Lambda 5}\left(\frac{3}{8\pi}\frac{M_p^2}{M_5^3} - 6r_c\right)}{9M_5^6\left(\frac{3}{8\pi}\frac{M_p^2}{M_5^3}\right)^2 - 2\alpha\rho_{\Lambda 5}\frac{3}{8\pi}M_p^2\left(\frac{3}{8\pi}\frac{M_p^2}{M_5^3} - 6r_c\right)}, \tag{3.13}$$

where M_p is the 4D Planck mass. In this case brane evolution equation (3.10) becomes:

$$H^2 + \frac{k}{a^2} = \frac{8\pi}{3M_p^2}\rho + V_1(\alpha, \rho_{\Lambda 5})\,\rho + \frac{8\pi}{3M_p^2}\rho_\Lambda, \tag{3.14}$$

where the (effective in this higher-dimensional model) 4D dark energy is:

$$\rho_\Lambda \equiv \rho_{\Lambda 4} = \frac{3M_p^2}{2\pi(\frac{1}{8\pi}\frac{M_p^2}{M_5^3} - 2r_c)^2} - \frac{M_p^2}{96\pi\alpha}\left(1 - \sqrt{1 + \tilde{\Lambda}}\right)\left(2 + \sqrt{1 + \tilde{\Lambda}}\right)^2. \tag{3.15}$$

In the equations above $\rho_{\Lambda 5}$ is the 5D bulk holographic dark energy, which according to (2.6) is given by:

$$\rho_{\Lambda 5} = c^2\frac{3}{4\pi}M_5^3 L^{-2}. \tag{3.16}$$

Relations (3.11)-(3.16) describe the low-energy (late-time) cosmological evolution on the brane. Similarly to [19, 20] the holographic nature of $\rho_{\Lambda 5}$ is the cause of the holographic nature of ρ_Λ. Finally, the 5D Planck mass M_5 is related to the standard 4D M_p through $M_5^3 = M_p^2/L_5$ (according to (2.2)), with L_5 the volume (size) of the extra dimension.

Let us make some comments here. The above expressions in the limit $\alpha \to 0$ (where $\tilde{\Lambda} \to 0$ and $V_1(\alpha, \rho_{\Lambda 5}) \to 0$) tend smoothly to those analyzed in [19]. However, in the presence of the Gauss-Bonnet term ($\alpha \neq 0$) we observe an interesting interacting behavior. Indeed, in (3.14) there is a coupling between ρ and $V_1(\alpha, \rho_{\Lambda 5})$, that is a term depending on $\rho_{\Lambda 5}$ and therefore on ρ_Λ. We mention that the coupling between ρ and ρ_Λ arises naturally through the full 5D dynamics and the use of bulk holographic dark energy, and it is not a result of an arbitrary introduction by hand, which is the usual case in interacting holographic dark energy in the literature [39, 47] even in the case where a Gauss-Bonnet term is present [48].

Our final goal is to find the relation between ρ_Λ and the metric scale factor a of the brane. However, the complex form of the above equations makes it impossible to acquire such an expression analytically. Therefore, in the following we describe the necessary approximations. Firstly, as we have already mentioned, according to (2.2) $M_5^3 = M_p^2/L_5$ with L_5 the volume of the extra dimension. In this work we assume that L_5 is arbitrary large (but not infinite), i.e. it is larger than any other length of the model, thus leaving brane evolution unaffected by the bulk size or bulk boundaries and this is the reason for the single-brane consideration. Therefore, in the calculations below we impose $M_p^2/M_5^3 = L_5 \gg r_c$ and $1/L_5 \to 0$. The role of the bulk size was investigated in [20]. Secondly, we expand (3.13) and (3.15) in terms of the Gauss-Bonnet coupling α and we keep only the linear term. Actually this is also a consistency requirement since, in heterotic string theory background, the Gauss-Bonnet form is the leading order quantum correction to gravity, i.e we have already kept only linear terms in α [49]. These steps lead to:

$$V_1(\alpha, \rho_{\Lambda 5}) \equiv V_1(\alpha, L) = \frac{4}{9}\frac{c^2}{M_p^2}\alpha L^{-2} + \mathcal{O}(\alpha^2), \tag{3.17}$$

$$\rho_\Lambda = 3c^2 \frac{1}{128\pi^2} M_p^2 L^{-2}\left(1 + \alpha \frac{c^2}{24\pi} L^{-2}\right) + \mathcal{O}(\alpha^2). \tag{3.18}$$

Finally, we have to determine the cosmological length L which is present in the bulk holographic dark energy expression (3.16) and has been transferred to relations (3.17),(3.18), too. In the

following we will consider a flat Universe, in order to safely use the future event horizon to define L, without entering into the relevant discussion of the literature concerning the IR cut-off in non-flat cases [1, 3, 4, 5, 40]. However, the model of the present work, such as the majority of braneworld models of the literature, is not maximally isotropic and this feature makes the analytical calculation of the future event horizon an impossible task. In this anisotropic case we can alteratively use the 4D future event horizon R_h (the 4D spacetime is the maximally isotropic subspace of the model), without losing the qualitative behavior of the observables. Fortunately, the calculations in the simple case without a Gauss-Bonnet term [19], showed that the use of the 4D future event horizon leads to identical quantitative results comparing to those obtained within the traditional holographic dark energy [1, 2, 3, 4, 5, 6].

Using the above approximations we obtain the following form for the effective 4D holographic dark energy:

$$\rho_\Lambda = 3c^2 \frac{1}{128\pi^2} M_p^2 R_h^{-2} \left(1 + \alpha \frac{c^2}{24\pi} R_h^{-2}\right), \tag{3.19}$$

and substitution to Friedmann equation (3.14), for the flat-Universe case, gives:

$$H^2 = \frac{8\pi}{3M_p^2} \rho \left(1 + \alpha \frac{c^2}{6\pi} R_h^{-2}\right) + \frac{c^2}{16\pi} R_h^{-2} \left(1 + \alpha \frac{c^2}{24\pi} R_h^{-2}\right). \tag{3.20}$$

In these relations, the 4D future event horizon R_h is given as usual by:

$$R_h = a \int_a^\infty \frac{da'}{Ha'^2}. \tag{3.21}$$

Finally, we have to insert in (3.20) the known form for $\rho(a)$, namely $\rho = \rho_0 a^{-3}$, with ρ_0 its present value.

The aforementioned integral equations determine completely the brane evolution, in the low energy limit, and up to first order in terms of the Gauss-Bonnet coupling α. In the limit $\alpha \to 0$ these expressions coincide with those extracted in [19]. However, in the presence of the Gauss-Bonnet term the implications are significant. Firstly, 4D holographic dark energy ρ_Λ, apart from the usual squared holographic term, acquires a quartic correction. Secondly, matter density ρ is coupled with a holographic term $\propto R_h^{-2}$, which is a result of ρ-ρ_Λ interaction of equation (3.14).

Analytical solution of equations (3.19)-(3.21), namely finding $H(a)$, then $R_h(a)$, and finally $\rho_\Lambda(a)$, is impossible. However, we are not interested in investigating the complete evolution but only in revealing the form of $\rho_\Lambda(a)$. Thus, we generalize Li's steps to construct a differential equation using Ω_Λ as the unknown function [1].

Firstly, we insert the usual variables: $\Omega_\Lambda = \frac{8\pi\rho_\Lambda}{3M_p^2 H^2}$, $\Omega_M = \frac{8\pi\rho}{3M_p^2 H^2}$. Relation (3.19) then gives:

$$R_h = \frac{c_1}{\sqrt{\Omega_\Lambda}H} + \alpha\, c_2 \sqrt{\Omega_\Lambda}H \tag{3.22}$$

up to $\mathcal{O}(\alpha^2)$, with $c_1 = \frac{c}{4\sqrt{\pi}}$ and $c_2 = \frac{c}{12\sqrt{\pi}}$. Inserting this form in (3.21) and using the variable $x = \ln a$ we obtain:

$$\int_x^\infty \frac{dx}{Ha} = \frac{1}{a}\left(\frac{c_1}{\sqrt{\Omega_\Lambda}H} + \alpha\, c_2 \sqrt{\Omega_\Lambda}H\right). \tag{3.23}$$

Similarly, using Ω_Λ, Ω_M, and R_h from (3.22), Friedmann equation (3.14) (with $V_1(\alpha, \rho_{\Lambda 5})$ given by (3.17)) up to $\mathcal{O}(\alpha^2)$ writes:

$$1 - \Omega_\Lambda = \Omega_M\left(1 + \alpha\, 2c_3\Omega_\Lambda H^2\right), \tag{3.24}$$

where $c_3 = 32\pi/3$. In order to proceed forward we have to assume an explicit $\Omega_M(a)$ dependence. In the interacting case at hand this should be different from the known $\sim a^{-3}$ behavior of standard cosmology. However, in our model the ρ-ρ_Λ interaction is downgraded by the extra-dimensional size as can be seen in (3.13) or equivalently in (3.17). Therefore, the deviation from conventional evolution will not be significant and we can use $\Omega_M = \Omega_M^0 H_0^2 H^{-2} a^{-3}$ with Ω_M^0 and H_0 the present values. Thus, we obtain:

$$\frac{1}{Ha} = \frac{\sqrt{a}\sqrt{1 - \Omega_\Lambda}}{\sqrt{\Omega_M^0}H_0}\left[1 - \alpha\, c_3\Omega_\Lambda\frac{\Omega_M^0 H_0^2}{a^3(1 - \Omega_\Lambda)}\right]. \tag{3.25}$$

Finally, substituting this relation to (3.23) and taking derivative with respect to x, up to $\mathcal{O}(\alpha^2)$ we acquire the following differential equation:

$$\Omega_\Lambda' = Q_1(\Omega_\Lambda) + \alpha\, Q_2(\Omega_\Lambda, a), \tag{3.26}$$

where

$$Q_1(\Omega_\Lambda) = \Omega_\Lambda^2(1 - \Omega_\Lambda)\left[\frac{1}{\Omega_\Lambda} + \frac{2}{c_1\sqrt{\Omega_\Lambda}}\right], \tag{3.27}$$

and

$$Q_2(\Omega_\Lambda, a) = \frac{\Omega_M^0 H_0^2}{c_1 a^3}\left\{(c_2 - c_3 c_1)\left[-5\Omega_\Lambda^2 + Q_1(\Omega_\Lambda)\left(\frac{1}{\Omega_\Lambda} - 1\right)^{-1}\right] - 2c_3\Omega_\Lambda^{5/2}\right\}, \tag{3.28}$$

and the prime denotes the derivative with respect to x. Note that in the limit $\alpha \to 0$, differential equation (3.26) tends smoothly to that obtain by Li in [1], namely $\Omega_\Lambda' = Q_1(\Omega_\Lambda)$, and can be

easily solved analytically. In the $\alpha \neq 0$ case of the present work such an exact solution is impossible. However, under the identification $\rho_\Lambda(a) \sim a^{-3(1+w_\Lambda)}$, we can extract the form of $w_\Lambda(z)$ at late times, i.e. at small z, with $z = \frac{a_0}{a} - 1$ and a_0 the value of a at present time (for simplicity we set $a_0 = 1$). We proceed as follows:

Firstly, expanding $\ln \rho_\Lambda$ we obtain:

$$\ln \rho_\Lambda = \ln \rho_\Lambda|_0 + \frac{d \ln \rho_\Lambda}{d \ln a}|_0 \ln a + \frac{1}{2} \frac{d^2 \ln \rho_\Lambda}{d(\ln a)^2}|_0 (\ln a)^2 + \mathcal{O}\left((\ln a)^3\right), \tag{3.29}$$

where the derivatives are calculated at the present time $a_0 = 1$ [1]. Therefore, through $\rho_\Lambda(a) \sim a^{-3(1+w_\Lambda)}$ we make the identification:

$$w_\Lambda = -1 - \frac{1}{3}\left[\frac{d \ln \rho_\Lambda}{d \ln a}|_0 + \frac{1}{2}\frac{d^2 \ln \rho_\Lambda}{d(\ln a)^2}|_0 \ln a + \mathcal{O}\left((\ln a)^2\right)\right]. \tag{3.30}$$

Now, using Friedmann equation (3.24), and the expressions $\Omega_\Lambda = \frac{8\pi\rho_\Lambda}{3M_p^2 H^2}$ and $\Omega_M = \Omega_M^0 H_0^2 H^{-2} a^{-3}$, we find:

$$\rho_\Lambda = \frac{3M_p^2 \Omega_\Lambda}{8\pi} \frac{\Omega_M^0 H_0^2}{a^3(1-\Omega_\Lambda)}\left[1 + \alpha\, 2c_3 \Omega_\Lambda \frac{\Omega_M^0 H_0^2}{a^3(1-\Omega_\Lambda)}\right], \tag{3.31}$$

up to $\mathcal{O}(\alpha^2)$. Therefore, differentiating this relation with respect to $\ln a = x$, and using (3.26) for the calculation of the derivatives, we finally obtain the following w_Λ expression:

$$w_\Lambda(z) = w_0 + w_1 z + \alpha(w_2 + w_3 z), \tag{3.32}$$

where

$$w_0 = -\frac{1}{3} - \frac{2}{3c_1}\sqrt{\Omega_\Lambda^0}, \tag{3.33}$$

$$w_1 = \frac{1}{6c_1}\sqrt{\Omega_\Lambda^0}(1-\Omega_\Lambda^0)\left(1 + \frac{2\sqrt{\Omega_\Lambda^0}}{c_1}\right), \tag{3.34}$$

$$w_2 = \frac{2}{3c_1}\frac{\Omega_\Lambda^0}{1-\Omega_\Lambda^0}\left[b_1 c_1 + 2b_2 b_3 c_1 - \sqrt{\Omega_\Lambda^0}(b_1 + b_2 b_3 - c_1 c_3 b_2)\right], \tag{3.35}$$

$$w_3 = -\frac{1}{6c_1^2}\frac{\Omega_\Lambda^0}{1-\Omega_\Lambda^0}\left\{-4(b_1 + 2b_2 b_3)c_1^2 + c_1(7b_1 + 15b_2 b_3 - 3b_2 c_1 c_3)\sqrt{\Omega_\Lambda^0} + \right.$$
$$+ (8b_2 c_1 c_3 - 6b_1 - 8b_2 b_3)\Omega_\Lambda^0 + c_1[b_1 - b_2(3b_3 + c_1 c_3)](\Omega_\Lambda^0)^{3/2} + $$
$$\left. + 2[b_1 + 2b_2(b_3 - c_1 c_3)](\Omega_\Lambda^0)^2\right\}. \tag{3.36}$$

In the expressions above we have used the constants $b_1 = 2c_3c_4$, $b_2 = c_4/c_1$ and $b_3 = c_2 - c_3c_1$, where $c_4 = \Omega_M^0 H_0^2$. Moreover, since $a_0 = 1$, we have replaced $\ln a = -\ln(1+z) \approx -z$. Finally, Ω_Λ^0 is the present value of Ω_Λ.

Relation (3.32) is the main result of this work and provides the Gauss-Bonnet correction to the corresponding result of [19]. Both investigations are formulated in the framework of bulk holographic dark energy. Therefore, although in the limit $\alpha \to 0$, (3.32) coincides with Li's expression in [1], namely $w_\Lambda(z) = w_0 + w_1 z$, the physical explanation in the present case comes through the 5D holographic consideration. This is the reason of the difference in constants between this work and [1].

From (3.32) it becomes obvious, that according to the value of c which is present in $\rho_{\Lambda 5}$-relation (3.16), of c_4 and of the Gauss-Bonnet coupling constant α, one can obtain a 4D holographic dark energy behaving as phantom [50], quintessence or quintom [21, 22], i.e crossing the phantom divide $w_\Lambda = -1$ [23, 43] during the evolution. Additionally, one can use observational results concerning dark energy evolution [24, 25] in order to estimate the bounds of the constant c of [1], i.e the bounds of c_1 of the present work. In particular, observational data from type Ia supernovae give the best-fit value $c_1 = 0.21$ within 1-σ error range [51], while those from the X-ray gas mass fraction of galaxy clusters lead to $c_1 = 0.61$ within 1-σ [52]. Similarly, combining data from type Ia supernovae, cosmic microwave background radiation and large scale structure give the best-fit value $c_1 = 0.91$ within 1-σ [53], while combining data from type Ia supernovae, X-ray gas and baryon acoustic oscillation lead to $c_1 = 0.73$ as a best-fit value within 1-σ [54]. Inserting this range of c_1 values into our model one finds that $w_0 < -1$ and $w_1 > 0$, thus, within 1-σ, he obtains a quintom-type holographic dark energy. Furthermore, $w_2 < 0$ while $w_3 > 0$ and therefore the Gauss-Bonnet contribution decreases the present value of w_Λ, while it increases the growing rate of $w_\Lambda(z)$ with z, in comparison with the case where such a term is absent. However, the quantitative correction of the $\alpha \neq 0$ case will be very small, for reasonable c_4 values. The reason is that, as we have mentioned, the ρ-ρ_Λ coupling, which arose naturally as a term $V_1(\alpha, \rho_{\Lambda 5}) \rho$ in (3.14), is downgraded by the extra-dimensional size as can be seen in (3.13) or equivalently in (3.17) (where we acquire a L^2 in the denominator). Thus, making the assumption that L_5 is arbitrary large we downgrade the Gauss-Bonnet correction, too. It should be interesting to investigate the case where the bulk-size is smaller than the future event horizon, as in the two-brane model of [20, 55], but with the inclusion of a Gauss-Bonnet

term. The subject is under investigation. Finally, note that the role of the Gauss-Bonnet term on the $w = -1$ crossing has been investigated both in conventional 4D [32] and in braneworld frameworks [29, 31]. The novel feature of our work is the combined investigation of such a term with the bulk holographic dark energy.

IV.   DISCUSSION-CONCLUSIONS

In this work we apply bulk holographic dark energy in a general braneworld model, with an induced gravity term and a perfect fluid on the brane, and a Gauss-Bonnet term in the bulk. Such a generalized bulk version of holographic dark energy is necessary if we desire to match the successes of brane cosmology in both theoretical and phenomenological-observational level, with the successful, simple, and inspired by first principles, notion of holographic dark energy in conventional 4D cosmology. In particular, as we showed in [19], the bulk space is the natural framework for the cosmological application, concerning dark energy, of holographic principle, since it is the maximally-dimensional subspace that determines the properties of quantum-field and gravitational theory, and the black hole formation. Subsequently, this bulk holographic dark energy will give rise to an effective 4D dark energy with "inherited" holographic nature, and this one will be present in the effective Friedmann equation.

Taking the Gauss-Bonnet combination into account, a ρ-ρ_Λ coupling appears in the Friedmann equation of the brane. We mention that this term arises naturally and is not a result of an inclusion by hand, which is the usual case of 4D interacting holographic dark energy in the literature [39, 47, 48]. This fact makes bulk holographic dark energy in the Gauss-Bonnet framework an interesting subject for further investigation.

Examining the low-energy (late-time) evolution of the aforementioned model, we acquire the relation of $w_\Lambda(z)$ up to $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(z^2)$. In the limit $\alpha \to 0$ we re-obtain the results of [19] and those of conventional 4D calculations [1, 2, 3, 4, 5, 6], although in the 5D study the interpretation and explanation of these results is fundamentally different. In the presence of Gauss-Bonnet combination, and taking into account the constraints on the values of the constants by observational data, we find that the effective 4D holographic dark energy behaves as a quintom, i.e it crosses the phantom divide $w_\Lambda = -1$ during the evolution. In particular, we observe that the presence of a non-zero α makes the current value of w_Λ smaller, while it

increases its growing rate with z, comparing to the $\alpha = 0$ case. However, the corresponding quantitative correction is very small due to the diminution of the ρ-ρ_Λ coupling by the arbitrary large extra-dimensional size. Yet, it should be interesting to investigate the case where the bulk size is smaller than the future event horizon. Then, the ρ-ρ_Λ coupling would be significant and we would naturally acquire the advantages of interacting holographic dark energy, such as the coincidence problem solution, and the corresponding effects on $w_\Lambda(z)$.

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